$\mathcal{C} \xrightarrow{\pi} h\mathcal{C} \implies$ isomorphisms in $\mathcal{C}$

quasicat — homotopy category

Functor category: $\mathcal{C}, \mathcal{D} \leftarrow$ categories

$\implies \mathrm{Fun}(\mathcal{C},\mathcal{D}) =$ obj: functors $\mathcal{C} \to \mathcal{D}$; mor: nat transf $\alpha : F \Rightarrow F' : \mathcal{C} \to \mathcal{D}$

$$\{A \times \mathcal{C} \dashrightarrow \mathcal{D}\} \longleftrightarrow \{A \dashrightarrow \mathrm{Fun}(\mathcal{C},\mathcal{D})\}$$

$X, Y \in sSet \implies \underset{sSet}{\mathrm{Fun}(X,Y)}$ – function complex

defined so $\mathrm{Fun}(X,Y)_n := \mathrm{Hom}(\Delta^n \times X, Y)$

$$\begin{array}{ccc} \mathrm{Fun}(X,Y)_n & := & \mathrm{Hom}(\Delta^n \times X, Y) \\ \uparrow & & \uparrow (\delta \times id_X)^* \\ \mathrm{Fun}(X,Y)_m & = & \mathrm{Hom}(\Delta^m \times X, Y) \end{array}$$

$\delta : [m] \to [n]$, $\Downarrow$, $\Delta^m \xrightarrow{\delta} \Delta^n$

Prop: $\mathrm{Fun} : sSet^{op} \times sSet \to sSet$ is a functor.

have:
$$\{K \times X \dashrightarrow Y\} \Longleftrightarrow \{K \dashrightarrow \mathrm{Fun}(X,Y)\}$$

$f : K \times X \to Y \implies \tilde{f} : K \to \mathrm{Fun}(X,Y)$

$k \longmapsto \tilde{f}(k) \in \mathrm{Fun}(X,Y)_n$, $k \in K_n$

$\tilde{f}(k) : \Delta^n \times X \to Y$

$\Delta^n \xrightarrow{k} K$

$\Downarrow$

$$\underbrace{\Delta^n \times X \xrightarrow{k \times id_X} K \times X \xrightarrow{f} Y}_{\tilde{f}(k)}$$

$sSet = \mathrm{Fun}(\Delta^{op}, Set)$

$$\implies \mathrm{Fun}(K, \mathrm{Fun}(X,Y)) \cong \mathrm{Fun}(K \times X, Y)$$

$sSet$ is cartesian closed: $X \times - : sSet \rightleftarrows sSet : \mathrm{Fun}(X,-)$

C, D quasicats $\Longrightarrow$ $\mathrm{Fun}(C,D) \in sSet$

__Claim:__ $\mathrm{Fun}(C,D)$ is a quasicat $\longleftarrow$

call it: __functor quasicategory__

$\hookrightarrow \begin{cases} \mathrm{Fun}(C,D)_0 = \text{functors} \\ \mathrm{Fun}(C,D)_1 = \text{nat. trans.} \end{cases}$

__Exc:__ C, D categories

$$N(\mathrm{Fun}(C,D)) \simeq \mathrm{Fun}(NC, ND)$$

functor category — function complex ✓

$$\begin{tikzcd} \Lambda^n_j \arrow[r] \arrow[d, hook] & C \\ \Delta^n \arrow[ur, dashed, "\exists"'] & \end{tikzcd} \qquad \begin{tikzcd} K \arrow[r] \arrow[d, hook] & C \\ L \arrow[ur, dashed, "?"'] & \end{tikzcd}$$

inner horns

__Weakly saturated class__ of morphisms in $sSet$ is A class of morphisms

such that

(1) A contains all isos ✓

(2) closed under cobase change

(3) ~~closed under composition~~

(4) closed under transfinite composition $\swarrow$

(5) ~~closed under coproducts~~

(6) closed under retracts

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & A' \arrow[d, "f'"] \\ B \arrow[r] & B' \end{tikzcd}$$

pushout, f' is "cobase change" of f

(2) $f \in A \Rightarrow f' \in A$

(3) $f, g \in A, \Rightarrow gf \in A$

(4) closed countable composits if $f_k \in A$, $k \in \mathbb{N} \Rightarrow \hat{f} \in A$

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \xrightarrow{f_3} X_3 \xrightarrow{f_4} \cdots \to \operatorname{colim}_n X_n = \hat{X}$$

st $X_0 \xrightarrow{\hat{f}} \operatorname{colim} X_n = \hat{X}$

closed under transfinite composition: λ = ordinal (= well-ordered set)

$X : \lambda \to sSet$ st $\forall\ i \in \lambda,\ i \neq 0,$

the map $\operatorname{colim}_{j<i} X(j) \to X(i) \in A$

$\Rightarrow \quad (X(0) \to \operatorname{colim}_{j<\lambda} X(j)) \in A$

---

(5) coproducts $\{X_i \to Y_i\} \subseteq A \Rightarrow (\coprod X_i \to \coprod Y_i) \in A$

(6) retracts: f is a retract of g if $\exists$ comm diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] \arrow[rr, bend left, "id_X"] & X' \arrow[r] \arrow[d, "g"] & X \arrow[d, "f"] \\ Y \arrow[r] \arrow[rr, bend right, "id_Y"'] & Y' \arrow[r] & Y \end{tikzcd}$$

closure: $g \in A \Rightarrow f \in A$

$f, g \in \text{ob}\ \text{Fun}([1], sSet)$

Prop: $\mathcal{C}$ = coll of simplicial cats

$A = \{ i : A \to B \mid \forall\ C \in \mathcal{C},\ \forall\ A \xrightarrow{f} C,$
$\exists$ extension $g : B \to C$, $g \circ i = f \}$

Then A is weakly saturated

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & C \\ B \arrow[ur, dashed, "\exists"'] \end{tikzcd}$$

prove this

Ex: $\mathcal{C} = qCats \Rightarrow A \supseteq \text{InnHorn}$

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"'] \arrow[rr, bend left] \arrow[dr, phantom, "p.o."] & A' \arrow[r, "f"] \arrow[d, "i'"] & C \\
B \arrow[r] \arrow[urr, dashed, "\exists"] & B' \arrow[ur, dashed, "\exists !"'] &
\end{tikzcd}$$

$i \in \mathcal{A}$ ✓ $\overset{??}{\Rightarrow}$ $i' \in \mathcal{A}$

have extension for maps to C

weakly cosaturated class
$=$
weakly sat class in opposite category.

$\text{InnHorn} := \{ \Lambda^n_k \subseteq \Delta^n, \ n \geq 2, \ 0 < k < n \}$

$\text{LHorn} := \{ \quad \cdot \quad n \geq 1, \ 0 \leq k < n \}$

$\text{RHorn} := \{ \quad \cdot \quad n \geq 1, \ 0 < k \leq n \}$

$\text{Horn} := \{ \quad \cdot \quad n \geq 1, \ 0 \leq k \leq n \}$

$S \subseteq \text{mor}(sSet) \Rightarrow \overline{S} :=$ "weak saturation of S"
$=$ smallest w.sat class containing S

$\overline{\text{InnHorn}} :=$ class of <u>inner anodyne</u> maps

$\Rightarrow$ If C quasicat, $(A \xrightarrow{i} B) \in \overline{\text{InnHorn}}$

$\Rightarrow \text{Hom}(B, C) \xrightarrow{i^*} \text{Hom}(A, C)$ is surjective.

<u>Rem</u>: Monomorphisms $\subseteq$ mor $sSet$ is weakly saturated.

$\text{InnHorn} \subseteq \overline{\text{InnHorn}} \subseteq$ Monomorphisms

---

<u>Example</u>: Generalized horn

$n \geq 1, \quad S \subseteq [n] = \{0, 1, 2, \ldots, n\}$

$$\Lambda^n_S := \bigcup_{i \in S} \Delta^{[n] \setminus i} \subseteq \Delta^n$$

$[n] \setminus i = \{0, \ldots, i-1, i+1, \ldots, n\}$

norm $\Lambda^n_k = \Lambda^n_{[n] \setminus k}$

Generalized inner horn:
Λ^n_S s.t. S is not an "interval"
i.e., $\exists \ s < t < s'$, $s, s' \in S$, $t \notin S$

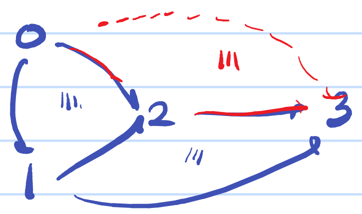
Lemma: any generalized inner horn $\Lambda^n_S \to \Delta^n$

is inner anodyne

Proof: app: Λ^3_2 $\subseteq$

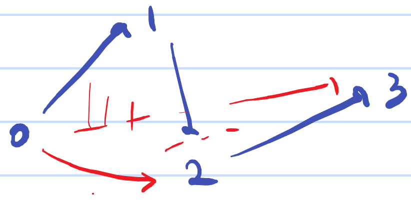
$\subseteq$

Ex: $\Lambda^3_{\{0,3\}} \subseteq \Delta^3$



$$\begin{tikzcd}
\Lambda^2_1 \cong \Lambda^{\{0,2,3\}}_{\{0,3\}} \arrow[r, tail, "\text{InnHorn}"'] \arrow[d] & \Delta^2 \cong \Delta^{\{0,2,3\}} \arrow[d] \\
\Lambda^3_{\{0,3\}} \arrow[r, tail, "f"] \arrow[r, "\text{InnHorn}"'] \arrow[rr, bend right, "\text{InnHorn}"'] & \Lambda^3_2 \arrow[r, tail, "\text{innHorn}"'] & \Delta^3
\end{tikzcd}$$

Lemma: $n \geq 2$, $\underset{\text{spine}}{I^n} \hookrightarrow \Delta^n$ is in $\overline{\text{InnHorn}}$



Lifting:

$$\begin{tikzcd} A \arrow[r, "?"] \arrow[d, "f"'] & X \\ B \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

"extension problem"

---

lifting problem for $f: A \to B$, $g: X \to Y$ is (u,v). st $vf = gu$

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r, "v"'] \arrow[ur, "s"] & Y \end{tikzcd}$$

lift is $s: B \to X$ st $sf = u$, $gs = v$

Lifting relation: write $f \boxslash g$, if a lift exists for any lifting problem (u,v) for (f,g)

"\boxslash"

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "f"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r] \arrow[ur, "s"] & \cdot \end{tikzcd}$$

equiv: $f \boxslash g$ iff $\quad f: A \to B$, $g: X \to Y$ $\quad$ "f lifts against g"

$$s \longmapsto (sf, gs)$$

$$\mathrm{Hom}(B,X) \longrightarrow \mathrm{Hom}(A,X) \times_{\mathrm{Hom}(A,Y)} \mathrm{Hom}(B,Y)$$

surjection

$$\left\{ \begin{tikzcd} A \arrow[r, "sf = u"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[ur, "s"] \arrow[r, "gs = v"'] & Y \end{tikzcd} \right\} \qquad = \qquad \left\{ (u,v) \;\middle|\; \begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r, "v"'] & Y \end{tikzcd} \right\}$$

$\mathcal{A}$ = class of maps

right complement $\mathcal{A}^{\boxslash} = \{ g \mid a \boxslash g \ \forall a \in \mathcal{A} \}$ $\Leftarrow$ weakly cosaturated

left complement ${}^{\boxslash}\mathcal{A} = \{ f \mid f \boxslash a \ \forall a \in \mathcal{A} \}$ $\Leftarrow$ weakly saturated

Exc: $\mathcal{A} \subseteq \mathcal{B} \implies \mathcal{A}^{\boxslash} \supseteq \mathcal{B}^{\boxslash} \implies {}^{\boxslash}(\mathcal{A}^{\boxslash}) \subseteq {}^{\boxslash}(\mathcal{B}^{\boxslash})$, with $\mathcal{A} \subseteq {}^{\boxslash}(\mathcal{A}^{\boxslash})$ and $\mathcal{B} \subseteq {}^{\boxslash}(\mathcal{B}^{\boxslash})$

$$\left({}^{\boxslash}(\mathcal{A}^{\boxslash})\right)^{\boxslash} = \mathcal{A}^{\boxslash}$$

Inner fibration: $\quad p: X \to Y$ if $\mathrm{InnHorn} \boxslash p$, where $\mathrm{InnHorn} = \{\Lambda^n_j \subset \Delta^n\}$

$\Updownarrow$

$\overline{\mathrm{InnHorn}} \boxslash p$

inner anodyne

$$\mathrm{InnFib} := \mathrm{InnHorn}^{\boxslash} = (\overline{\mathrm{InnHorn}})^{\boxslash}$$

Ex: $C \xrightarrow{p} * = \Delta^0$ is an inner fibration iff C is a quasicat

$C \in sSet$

$$\begin{tikzcd} \Lambda^n_j \arrow[r] \arrow[d, hook] & C \arrow[d, "p"] \\ \Delta^n \arrow[r] \arrow[ur, dashed, "?"] & * \end{tikzcd}$$

C, D ord categories $\subset$ (nerve) quasicats

$f: C \to D$ functor

Q: When is f an inner fibration

A: Always (exc)

$C' \overset{i}{\hookrightarrow} C$ subcpx, C', C quasicats

$i \in \mathrm{InnFib} \iff C'$ is a subcategory of C.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] \arrow[dr, phantom, "\ulcorner"] & X \arrow[d] \\ B \arrow[r] & Y \end{tikzcd}$$

$$\begin{tikzcd} A_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & A_1 \arrow[d] \\ A_2 \arrow[r] & A \end{tikzcd} \qquad \begin{tikzcd} B_0 \arrow[r] \arrow[d] & B_1 \arrow[d] \\ B_2 \arrow[r] & B \end{tikzcd}$$

$$\begin{tikzcd} A_1 \arrow[d, "i_1"'] & A_0 \arrow[l] \arrow[r] \arrow[d, "i_0"'] & A_2 \arrow[d, "i_2"'] \\ B_1 & B_0 \arrow[l] \arrow[r] & B_2 \end{tikzcd} \xrightarrow{\text{colim}} \begin{tikzcd} A \arrow[d, "i"] \\ B \end{tikzcd}$$

$i_1, i_0, i_2 \in \mathcal{A} \implies i \overset{?}{\in} \mathcal{A}$ <u>**NO**</u>

<u>Idea:</u>

If B = weakly sat, then $B = {}^{\boxtimes}(B^{\boxtimes})$